CS 4510 Automata and Complexity January 11th 2023

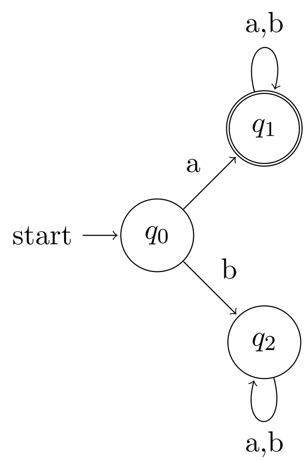
Lecture 2: Nondeterminism

Lecturer: Abrahim Ladha *Scribe(s): Samina Shiraj Mulani*

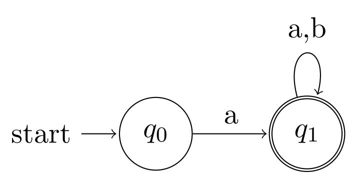
1 Introduction

We noted that DFAs are weak. Let's try to extend or generalize them. A DFA can be represented as $(Q, \Sigma, \delta, q_0, F)$. When thinking about extending DFAs, the only useful thing to extend is the way in which states interact with each other, i.e., δ.
Let's extend δ in 3 ways:

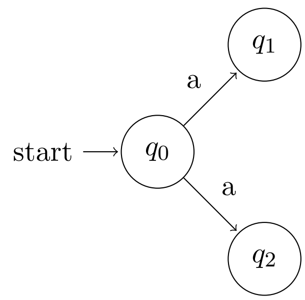
1. If a transition is undefined, we implicitly reject. As an example, Consider the following DFA which represents the language $\{w \in \Sigma^* \mid w \text{ begins with } a\}$



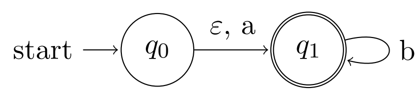
We can now represent this as



2. Allow transitions of more than one of the same type. This means that you can have multiple outgoing transitions with the same input. For example



3. Allow "ε-transitions", which can be taken for free. For example



a, ab, abb, b, bb, ε are some of the strings accepted by this NFA.

2 Coping with nondeterminism

Its important to understand nondeterminism and not just have deterministic coping strategies. We say a nondeterministic computation accepts if at least one computation path reaches an accept state. The following analogies help in visualizing this aspect -

1. **Depth First Search** DFS on the DFA until you reach an accept state.
2. **Lucky Coin** Imagine you flip a lucky coin that tells you exactly which fork/path to take.
3. **Alternate timelines** For each nondeterministic action, create multiple timelines, similar to creating multiple copies.

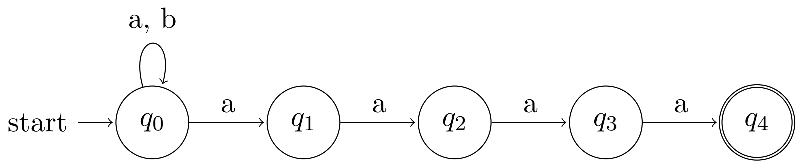
3 Definition and a few examples

A Nondeterminisitic Finte Automata (NFA) can be represented by a 5-tuple $(\Sigma, Q, q_0, \delta, F)$ where:

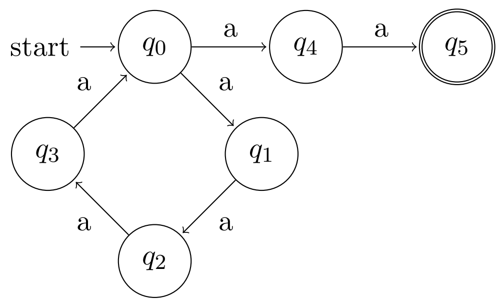
1. Σ - finite alphabet
2. Q - finite set of states
3. q_0 - denoted start state
4. δ - $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$
   $\mathcal{P}(Q)$ represents the power set of Q, which is the set of all subsets of Q. The power set of a set Q has 2^n elements where n is the number of elements in Q.
5. F - the set of final states $F \subseteq Q$

Lets show a few examples

1. $L_1 = \{w \in \Sigma^* \mid w \text{ ends with } aaaa\}$



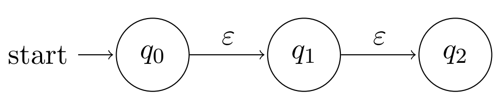
2. $L_{x,y} = \{a^{xn+y} \mid n \in \mathbb{N}\}$
   Lengths of the strings in this language form an Arithmetic Progression. We can show that there exists an NFA for every x, y. Note that the loop is of length x while the tail (q_4 to q_5 in the representation) is of length y.



Every DFA is an NFA, i.e., $\mathscr{L}(DFA) \subseteq \mathscr{L}(NFA)$. For every NFA, we will show you can construct an equivalent DFA. This means that $\mathscr{L}(NFA) \subseteq \mathscr{L}(DFA)$. Combining the aforementioned point, we get $\mathscr{L}(DFA) = \mathscr{L}(NFA)$ Note that when we perform a complement of the accept states in a DFA that represents a language L, we get the complement of the language. The same doesn't hold for an NFA due to the presence of the implicit reject state.

4 $\mathscr{L}(NFA) \subseteq \mathscr{L}(DFA)$

We simulate an NFA on a DFA. Each state of the DFA corresponds to any number of states of the NFA. Sure, a NFA can be in many states, but only finitely many. To each possible set of states, we will simulate that on our DFA as a single state. We first define the concept of reach.
$reach(q_i) = \{q_i \text{ and any state reachable from } q_i \text{ by } \varepsilon \text{ transitions}\}$. For example

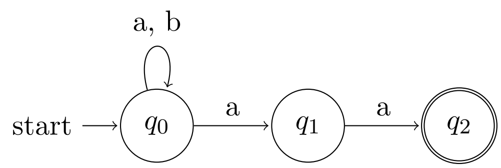


Then $reach(q_0) = \{q_0, q_1, q_2\}$. Let the NFA $N = (\Sigma, Q, q_0, \delta, F)$ Make $D = (\Sigma', Q', q_0{}', \delta', F')$

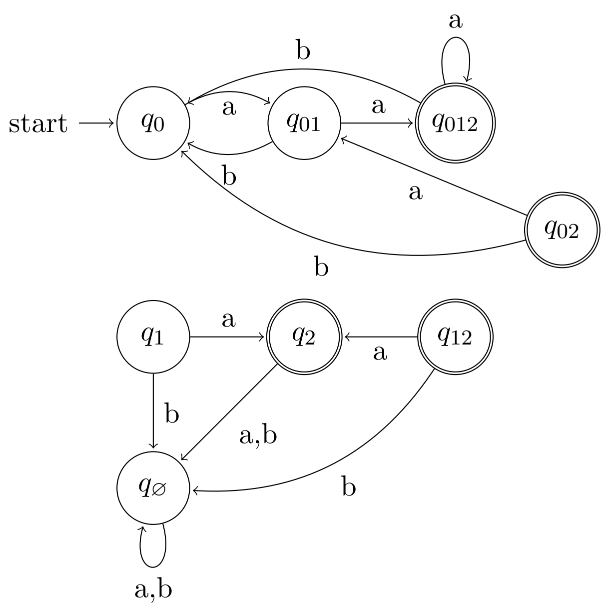
- $\Sigma' = \Sigma$
- $Q' = \mathcal{P}(Q)$

- $q_0{}' = reach(q_0)$
- $\delta'(\{q_1, ..., q_k\}, a) = \bigcup_{i=1}^{k} reach(\delta(q_i, a))$
- $F' = \{f \subseteq Q \mid f \cap F \neq \varnothing\}$

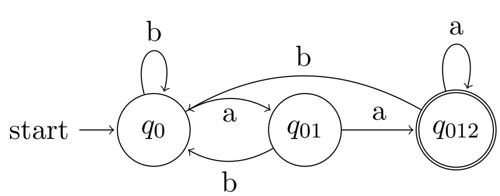
Example -
$L_2 = \{w \in \Sigma^* \mid w \text{ ends with } aa\}$



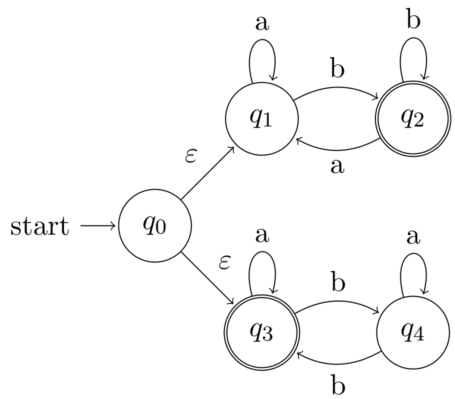
By following the above algorithm, we get the corresponding DFA



We observe that there are unreachable states (example - q_{02}). So the algorithm may or may not give a minimal DFA. On cleaning up these unreachable states, we get the following DFA



Each state represents a superposition of the states in the NFA. One utility of NFAs is that we can use them to create a more convenient representation of the union of two languages. Consider $L_3 = \{w \in \Sigma^* \mid w \text{ ends with } b\}$ and $L_4 = \{w \in \Sigma^* \mid \#b(w) \text{ is even}\}$ We can construct the following NFA that represents $L_3 \cup L_4$



We can also use this idea to prove that the union of 2 regular languages is always regular. An alternate is to follow last lecture's approach using Cartesian Product, which can be comparatively more cumbersome.

5 The Road Not Taken

By Robert Frost, emphasis mine

Two roads diverged in a yellow wood,
And sorry **I could not travel both**
And **be one traveler,** long I stood
And looked down one as far as I could
To where it bent in the undergrowth;
Then took the other, as just as fair,
And having perhaps the better claim,
Because it was grassy and wanted wear;
Though as for that the passing there
Had worn them really about the same,
And both that morning equally lay
In leaves no step had trodden black.
Oh, I kept the first for another day!
Yet knowing how way leads on to way,
I doubted if I should ever come back.
I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I—
I took the one less traveled by,
And that has made all the difference.

The moral of this poem in the context of our lecture is that Robert Frost is a deterministic actor, one who sees two roads and is forced to choose. If he was nondeterministic, he wouldn't have to choose. He could come to a fork in the road and just take it.