CS 4510 Automata and Complexity January 12, 2023

Lecture 2: Nondeterminism

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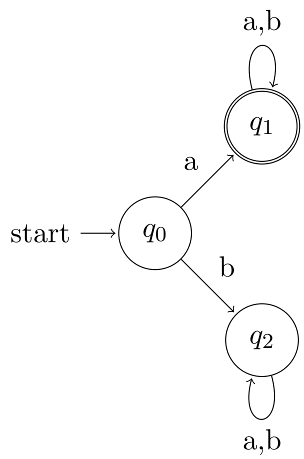
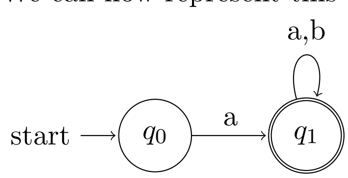
1 Introduction

We noted that DFAs are weak. Let's try to extend or generalize them.
A DFA can be represented as $(Q, \Sigma, \delta, q_0, F)$. When thinking about extending DFAs, the only useful thing to extend is the way in which states interact with each other, i.e., δ.
Let's extend δ in 3 ways:

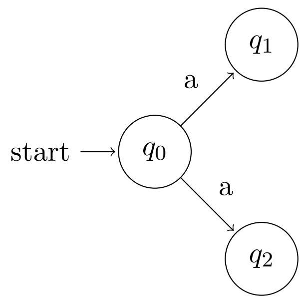
1. If a transition is undefined, we implicitly reject.
   As an example,
   Consider the following DFA which represents the language
   $\{w \in \Sigma^* \mid w \text{ begins with } a\}$ -

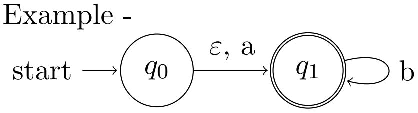
   

   We can now represent this as -

2. Allow transitions of more than one of the same type. This means that you can have multiple outgoing transitions with the same input.
   Example -



3. Allow "ε-transitions", which can be taken for free.
   Example -
   

   a, ab, abb, b, bb, ε are some of the strings accepted by this NFA.

2 Coping with nondeterminism

Its important to understand nondeterminism and not just have deterministic coping strategies.
We say a nondeterministic computation accepts if at least one computation path reaches an accept state.
The following ways help in visualizing this aspect -

1. DFS
   DFS on the DFA until you reach an accept state.

2. Lucky Coin
   Imagine you flip a lucky coin that tells you exactly which fork/path to take.

3. Alternate timelines
   For each nondeterministic action, create multiple timelines, similar to creating multiple copies.

3 Definition and a few examples

A Nondeterminisitic Finte Automata (NFA) can be represented by a 5-tuple $(\Sigma, Q, q_0, \delta, F)$ where:

1. Σ - finite alphabet

2. Q - finite set of states

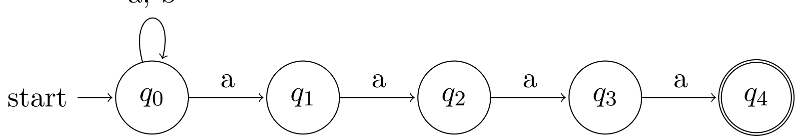
3. q_0 - denoted start state

4. δ - $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$
   $\mathcal{P}(Q)$ represents the power set of Q, which is the set of all subsets of Q. The power set of a set Q has 2^n elements where n is the number of elements in Q.

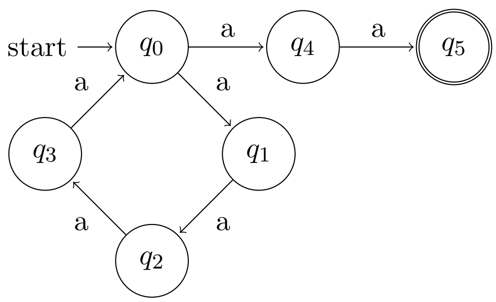
5. F - the set of final states F $\subseteq$ Q

A few examples -

1. $L_1 = \{w \in \Sigma^* \mid w \text{ ends with } aaaa\}$



2. $L_{x,y} = \{a^{xn+y} \mid n \in \mathbb{N}\}$
   Lengths of the strings in this language form an Arithmetic Progression. We can show that there exists an NFA for every x, y.
   Note - the loop is of length x while the tail (q_4 to q_5 in the representation) is of length y.



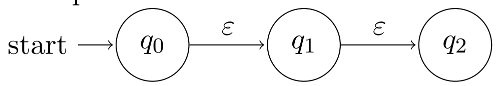
- Every DFA is an NFA, i.e., $\mathscr{L}(DFA) \subseteq \mathscr{L}(NFA)$.
- For every NFA, we can construct an equivalent DFA (see next section). This means that $\mathscr{L}(NFA) \subseteq \mathscr{L}(DFA)$.
  Combining the aforementioned point, we get $\mathscr{L}(DFA) = \mathscr{L}(NFA)$
- Note that when we perform a complement of the accept states in a DFA that represents a language L, we get the complement of the language. The same doesn't hold for an NFA due to the presence of the implicit reject state.

4 $\mathscr{L}(NFA) \subseteq \mathscr{L}(DFA)$

We simulate an NFA on a DFA. Each state of the DFA corresponds to any number of states of the NFA.
We first define the concept of reach -

$reach(q_i) = \{q_i \text{ and any state reachable from } q_i \text{ by } \varepsilon \text{ transitions}\}$

Example -



$reach(q_0) = \{q_0, q_1, q_2\}$

Let the NFA $N = (\Sigma, Q, q_0, \delta, F)$

Make $D = (\Sigma', Q', q_0', \delta', F')$

$\Sigma' = \Sigma$

$Q' = \mathcal{P}(Q)$

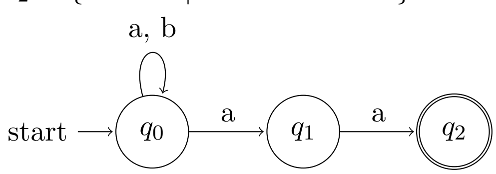
$q_0' = reach(q_0)$

$\delta'(\{q_1, ..., q_k\}, a) = \cup_{i=1}^{k} reach(\delta(q_i, a))$

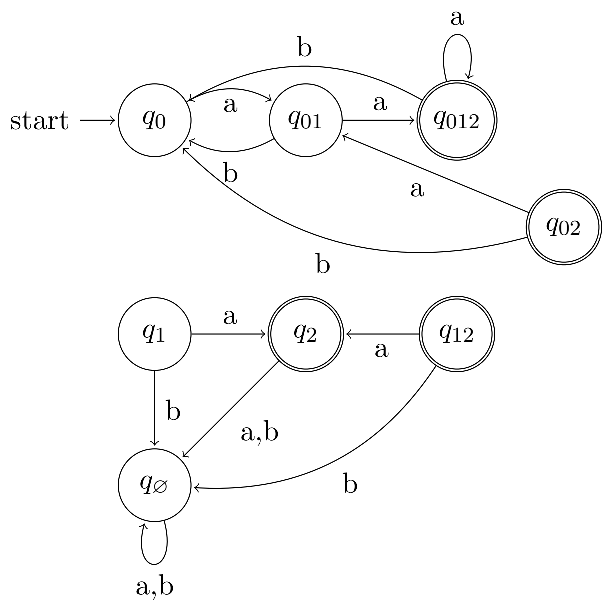
$F' = \{\rho \subseteq Q \mid \rho \cap F \neq \varnothing\}$

Example -

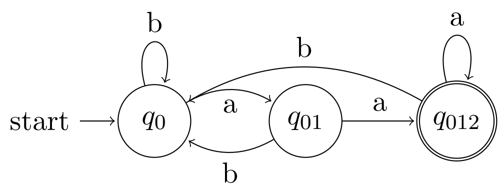
$L_2 = \{w \in \Sigma^* \mid w \text{ ends with } aa\}$



By following the above algorithm, we get the corresponding DFA -

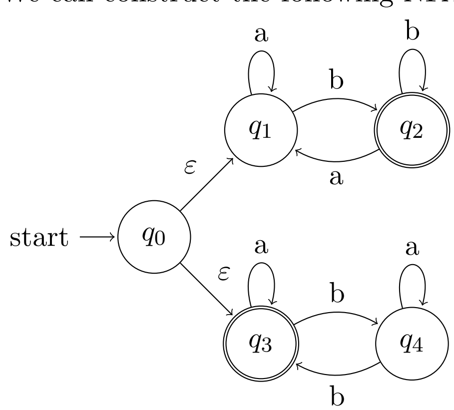


We observe that there are unreachable states (example - q_{02}). So the algorithm may or may not give a minimal DFA. On cleaning up these unreachable states, we get the following DFA -



Each state represents a superposition of the states in the NFA.

A utility of NFA is that we can use them to create a more convenient representation of the union of two languages.
Consider
$L_3 = \{w \in \Sigma^* \mid w \text{ ends with } b\}$
$L_4 = \{w \in \Sigma^* \mid \#b(w) \text{ is even}\}$
We can construct the following NFA that represents $L_3 \cup L_4$ -



We can also use this idea to prove that the union of 2 regular languages is always regular. An alternate is to follow last lecture's approach using Cartesian Product, which can be comparatively more cumbersome.

5 The Road Not Taken

By Robert Frost, emphasis mine

Two roads diverged in a yellow wood,
And sorry **I could not travel both**
And **be one traveler,** long I stood
And looked down one as far as I could
To where it bent in the undergrowth;
Then took the other, as just as fair,
And having perhaps the better claim,
Because it was grassy and wanted wear;
Though as for that the passing there
Had worn them really about the same,
And both that morning equally lay
In leaves no step had trodden black.
Oh, I kept the first for another day!
Yet knowing how way leads on to way,
I doubted if I should ever come back.
I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I—
I took the one less traveled by,
And that has made all the difference.

The moral of this poem in the context of our lecture is that Robert Frost is a deterministic actor, one who sees two roads and is forced to choose. If he was nondeterministic, he wouldn't have to choose. He could come to a fork in the road and just take it.