Linear Programming

example
feasible
standard form
shortest paths
knapsack
Max flow
polytopes
simplex
example.
reduction from
VC

Linear programming is a class of problem in which you have some (linear) set of constraints and multiple variables, and a objective function. The goal is to find a solution which is maximum.

Suppose you sell things. Thing 1 sells for 1 buck each and thing 2 sells for 6 bucks each, thing 2 is a luxury idem.
Your factory can only currently produce 400 items a day.
you also cannot make more than 200 of x_1 or 300 of x_2 due to sanctions
we may present this as the following LP

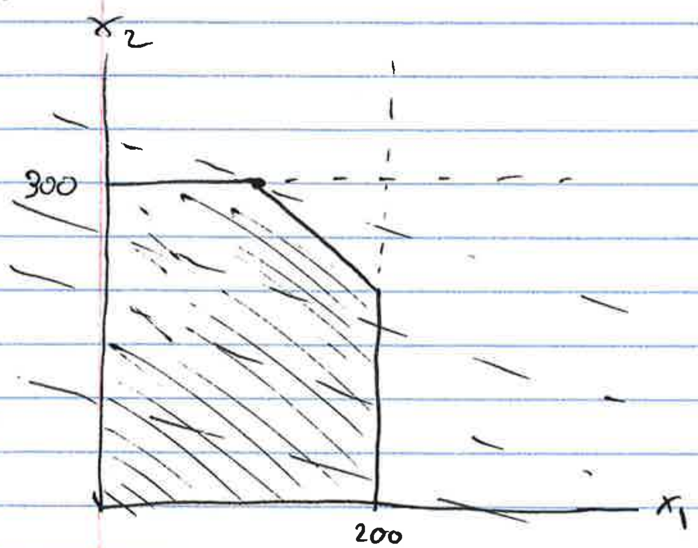
OP: $\max\ x_1 + 6x_2$

subject to:
$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$

$$c^T = [1, 6] \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix} b$$

LP
OP
MIP
ILP

we we to plot these inequalities and intersect the values which they are true for it would look like. our max function is



optimal is unachievable only when

1 infeasible $x \ge 1$ $x \le -1$

2 unbounded $\max\ x_1 + x_2$

$x_1, x_2 \ge 0$

standard form, want solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\max\ c^T x$

subj to $Ax \le b$

$x \ge 0$

can transform many things into standard form.

Suppose in G we want to compute shortest s-t path ~~&~~ let d_v be the shortest path from s to v. note $d_s = 0$.
we may add constraints $d_v \le d_u + w(u,v)$
for each u, v in G which share an edge.

$$\max d_t$$
$$\text{subj } d_v \le d_u + w(u,v) \quad \forall (u,v) \in E$$
$$d_s = 0$$

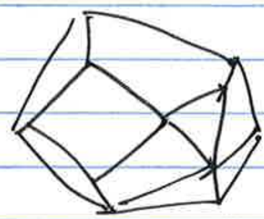
why maximize? $d_t = 0$ is min solution, but rest of constraints have
$d_t = \min_{u,(u,t)\in E} \{d_u + w(u,t)\}$ need to enforce that d_t is actually the path! d_t is largest value $\le$ all in this set.

max flow
Given flow network G where edges have capacities $c(u,v) \ge 0$
we want to maximize the flow from s to t

max flow leaving s - flow into source $= \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s)$
subject to $f(u,v) \le c(u,v)$
$f(u,v) \ge 0$
$\sum_{v\in V} f(v,u) = \sum_{v\in V} f(u,v)$ conservation

Knapsack items (v_i, w_i) 01 knapsack is a 01 program.

$$\max \sum v_i x_i$$
subject to $\sum w_i x_i < W$ are constraint!
$0 \le x_i \le 1$



heres our algorithm, simplex. Think of it as traversal over the polytope of the feasible region

begin $v = (0,0)$

while True

~~for each neighbor v' of v:~~

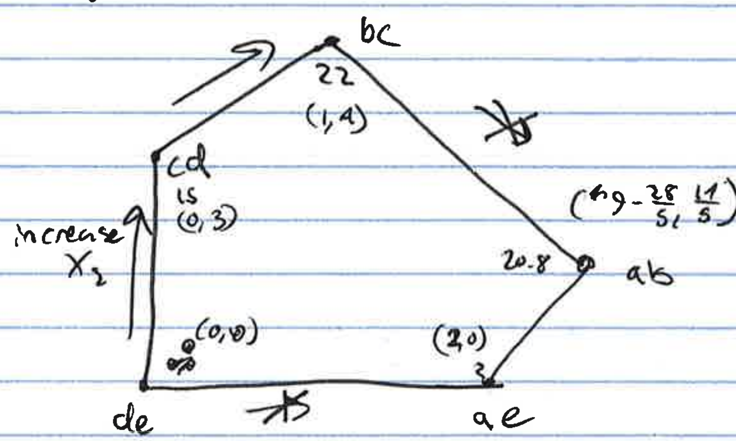
if v' neighbor of v is better

$v = v'$

else

break

determining v' easy at origin repeatedly change the coordinates!



$$\max \; 2x_1 + 5x_2$$

subj to

$$2x_1 - x_2 \le 4 \quad (a)$$
$$x_1 + 2x_2 \le 9 \quad (b)$$
$$-x_1 + x_2 \le 3 \quad (c)$$
$$x_1 \ge 0 \quad (d)$$
$$x_2 \ge 0 \quad (e)$$

release e until c is tight. $(0,0) \to (0,3)$ 15 vs 2

release c, d until c, b is tight $(0,3) \to (1,4)$ $2 \cdot 1 + 5 \cdot 4 = 22$

$2x_1 - x_2 \le 4$ $\quad x_1 \le 9 - 2x_2$

$x_1 + 2x_2 \le 9$ $\quad 2(9-2x_2) - x_2 \le 4$

$18 - 4x_2 - x_2 \le 4$

~~$2x_1 - x_2 \le 4$~~ $\quad 18 - 5x_2 \le 4$

$-x_1 + 3x_2 \le 5$ $\quad -5x_2 \le 4$

$0 + 5x_2 \le \cancel{5+9}\ 14$

$x_2 = \frac{14}{5}$ $\quad x_1 + 2 \cdot \frac{14}{5} \le 9$

$x_1 \quad 9 - \frac{2 \cdot 14}{5}$

$18 - \frac{4 \cdot 14}{5} + 14$

$32 - \frac{56}{5}$

ILP is NP-complete   decision variant check if x-treers satisfies equations and max > something $\{A, b, c, t \mid \exists x [Ax \le b] \; c^T x \ge t\}$

$VC \le ILP$ for edge u, v add variables x_u, x_v
and $x_u + x_v \ge 1$

$VC = \{\langle G, g\rangle \mid S \subset V$ every edge touches one ~~of one~~ of S $|S| = g\}$