1

Duality

11/28

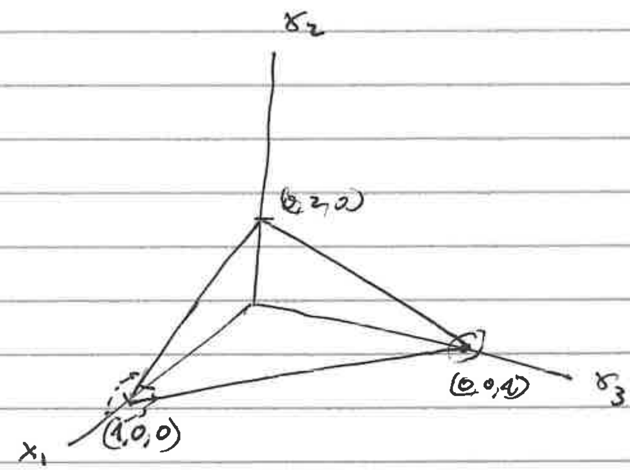
Last time we discussed what linear programming was and how many diverse problems it can represent. Today we do a far more in depth example and detail the simplex method, as well as the duality theorems

Recall LP standard form. Given $\underset{m\times n}{A}$ $\underset{m\times 1}{b}$ $\underset{1\times n}{c}$ we see a solution $\underset{1\times 1}{x}$ such that

$$\max c^T x$$
$$\text{subject to } Ax \le b$$
$$x \ge 0.$$

consider the following LP:

$$\max \; 2x_1 + 3x_2 + 5x_3$$
$$\text{subject to } \; x_1 + 2x_2 + x_3 = 4$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$
$$x_3 \ge 0$$



each constraint adds a geometric figure to our n space. The set of feasible solutions to an LP, forms a polyhedron in n space. the constraints $x_1, x_2, x_3 \ge 0$ force us into the positive octant. This forms a triangle based pyramid! In order to maximize $c^T x$, we take its plane an intersect it with our geometric figure. On some fixed plane the points will be equal so we seek ~~as~~ a maximal place to shift the plane given its slope. It is a guarantee that such a maximum exists at an extremal point, or ~~anot~~ (equal to all points on a line). The simplex algorithm simply iterates over the polyhedron's extremal points increasing the objective function each time until it cannot be increased any more. Here we see the result is ~~(4,0,0)~~ (0,0,4) with a max of 20.

It is a feat we are searching for a real number solution, but need only check finitely many extremal points.

Lets now do a more complex example. First we put our LP into "slack form" to convert inequalities to equalities

$$\max\ 3x_1 + x_2 + 2x_3$$
$$\text{subj to } x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 \quad 2x_2 \quad 5x_3 \le 24$$
$$4x_1 \quad x_2 \quad 2x_3 \le 36$$
$$x_1, x_2, x_3 \ge 0$$

$$z = 3x_1 + x_2 + 2x_3$$
$$s_4 = 30 - x_1 - x_2 - 3x_3 \quad (a)$$
$$s_5 = 24 - 2x_1 - 2x_2 - 5x_3 \quad (b)$$
$$s_6 = 36 - 4x_1 - x_2 - 2x_3 \quad (c)$$

sol = (0, 0, 0, 30, 24, 36) so $z = 0$.

if $Ax \le b$, then have s so $Ax + s = b$. Then slack form is just $s = b - Ax$. Denote those on the LHS as basic variables and those on the RHS as nonbasic. As we run simplex, we will "pivot". This will rewrite our LP swapping one basic for one non basic to maximize the objective function. Think of basic variables as those which are loose. An equality constraint is tight if its non basic variables get its basic variable to be $= 0$.
choose variable in z whose coefficient is largest and is nonbasic. This is the entering variable. Choose the tightest constraint, that basic variable will leave. So we swap x_1, x_6. ~~we~~ ~~we~~ rewrite c in x_1

$$x_1 = \cancel{9 - \frac{x_2}{4} - \frac{x_3}{2}} \qquad 9 - x_2/4 - x_3/2 - x_6/4$$

~~we~~ rewrite the LP substituting this in. we are basically doing gaussian elimination

$$z = 27 + x_2/4 + x_3/2 - 3x_6/4$$
$$x_1 = 9 - x_2/4 - x_3/2 - x_6/4$$
$$x_4 = 21 - 3x_2/4 - 5x_3/2 + x_6/4$$
$$x_5 = 6 - 3x_2/2 - 4x_3 + x_6/2$$

sol = (9, 0, 0, 21, 6, 0)
$z = 27$

now what variable do we choose? not x_6, its coefficient is now negative. x_3 is largest nonbasic. tightest constraint is x_5. You choose tightest constraint by seeing which one limits increase of ~~x_6~~ ~~the~~ x_3 the most

$$x_3 = 3/2 - \frac{3x_2}{8} - x_5/4 + x_6/8$$

substitute into get

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11}{16}x_6 \qquad sol = (33/4,\ 0,\ 3/2,\ 69/4,\ 0,\ 0)$$
$$x_1 = 33/4 - x_2/16 + x_5/8 - \frac{5x_6}{16} \qquad z = 111/4. \quad 27.75$$
$$x_3 = 3/2 - \frac{3}{8}x_2 - x_5/4 + x_6/8$$
$$x_4 = 69/4 + \frac{3x_2}{16} + 5x_5/8 - x_6/16$$

increase x_2, the upperbounds are $132, 4, \infty$. x_2 increases x_4, choose second equation
x_2 enters, x_3 leaves.

$$z = 28 - x_3/6 - x_5/6 - 2x_6/3 \qquad sol = (8, 4, 0, 18, 0, 0)$$
$$x_1 = 8 + x_3/6 + x_5/6 - x_6/3 \qquad z = 28.$$
$$x_2 = 4 - 8x_3/3 - 2x_5/3 + x_6/3$$
$$x_4 = 18 - x_3/2 + x_5/2 \quad 0$$

since all coeffs in z are negative then we cannot continue, $z = 28$ optimal.
slack variable solutions determine how much slack was in each original one.

Duality: ~~Re~~ How do we know simplex returns the optimum.? Recall how a max-flow mincut solution was optimum. If there was a mincut = max flow it was optimum. similarly, take an LP in standard form. It has an evil LP called a Dual ~~[illegible]~~ which minimizes. Their solutions are the same.

Primal	Dual
$\max\ c^T x$	$\min\ b^T y$
subj $Ax \le b$	subj $A^T y \ge c$
$x \ge 0$	$y \ge 0.$

note that the Dual of the Dual is the primal. They are brothers.

<u>weak duality</u> Theorem. Given a primal and its dual, we prove

$$c^T x \leq b^T y$$

The strong duality theorem will show these are equal. This is analogous to showing max flow ≤ min cut.

Proof:

$$c^T x = \sum_{j=1}^{n} c_j x_j \overset{A^T y \geq c}{\leq} \sum_{j=1}^{n}\left[\sum_{i=1}^{m}(a_{ij} y_i) x_j\right] = \sum_{i=1}^{m}\left[\sum_{j=1}^{n}(a_{ij} x_j) y_i\right] \overset{Ax \leq b}{\leq} \sum_{i=1}^{m} b_i y_i = b^T y$$

□

① If some $c^T x = b^T y$ then x, y are optimal for their LP's.

② P unbounded ⇒ D infeasible

D unbounded ⇒ P infeasible

Why is simplex optimal? It actually solves both the primal and the dual simultaneously. ~~we chose certain~~ just like how maxflow alg also finds the min cut.

Recall our solution was $(x_1, x_2, x_3) = (8, 4, 0)$ with $z = 28$. with

$z = 28 - x_3/6 - x_5/6 - 2x_6/3 = v' + \sum c_j' x_j$ then

$$y_i = \begin{cases} -c'_{n+i} & \text{is } x_{n+i} \text{ nonbasic} \\ 0 & \text{otherwise.} \end{cases}$$

the final objective function of the dual is then

$y_1 = 0$, $y_2 = 1/6$, $y_3 = 2/3$. Plugging this into $b^T y$ we get 28 as well.

How to give a certificate of optimality:

$$\max \; x_1 + 6x_2$$
$$x_1 \le 200 \quad (a)$$
$$x_2 \le 300 \quad (b)$$
$$x_1 + x_2 \le 400 \quad (c)$$

simplex returns $x_1 = 100$, $x_2 = 300$ with max $= 1900$. use gaussian elimination! $5b + c = x_1 + 6x_2 \le 1500 + 400 = 1900$.
so the max of the objective function cannot exceed 1900 so it is optimum

run time
avg case is polynomial, read about smooth analysis.
each iteration is $O(\cancel{\ } mn)$, but possibly $\binom{m+n}{n}$ iterations, exponential.