CS 4510 Automata and Complexity October 2nd 2023

Lecture 10: Turing Machines

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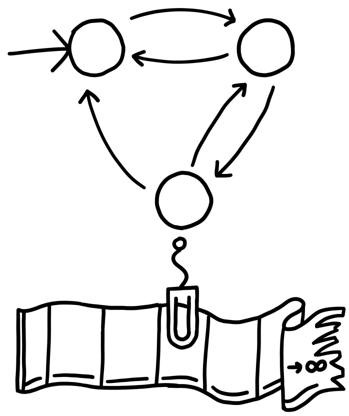
Turing machines are so interesting, every lecture for the remainder of the course will be about Turing machines. I want this or that interesting sub-topic to be its own lecture, so that leaves us with nothing for today except definitions and programming. Recall the limitations of a PDA. You pop something out the stack, its gone into the ether, forever. What if we could iterate over our memory structure in a non destructive way. Our motivation is then to just give a DFA a tape.



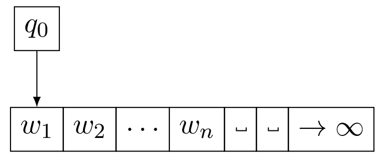
1 Definitions

A Turing machine is a tuple: $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ where:

- Q : finite set of states
- Σ : finite input alphabet
- Γ : finite tape alphabet
  - $\textvisiblespace \in \Gamma$, this is the symbol for a blank space on the tape[1]
  - $\textvisiblespace \notin \Sigma$
  - $\Sigma \subsetneq \Gamma$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\text{L,R}\}$ is our (deterministic) transition function.
- $q_0 \in Q$: denoted start state
- $q_a, q_r \in Q$: denoted accept and reject states

[1]For latex, you may use either `textvisiblespace` or `sqcup`

There is an immediate difference between the different machines we have seen so far. The Turing machine as defined does not have a separate part for it to read the input. A Turing Machine is initialized with $w_1 w_2 ... w_n$ on the leftmost cells of the tape, and the tapehead on w_1. All other cells initialize to $\llcorner\!\lrcorner$.

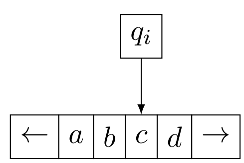


The Turing machine also need not be forced to accept when it has fully read the input. Instead of accepting/rejecting being determined by whatever the last state you land in, the states q_a, q_r are more like instructions than states. Upon entering them, all computation is halted and there are no more outgoing transitions. Turing machines may even get stuck in infinite loops. If the machine attempts to move left past the first cell, then it will simply be reset back to the first cell. By convention, we usually let $\Gamma = \Sigma \cup \{\llcorner\!\lrcorner\}$.

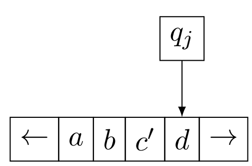
1.1 Configurations

A configuration of a Turing machine is a string encoding of an instantaneous description, or snapshot, of a Turing machine. We say a configuration C yields C' if C' follows C after one step of the transition function.. We write this as $C \vdash C'$. Note that since the machine is deterministic, given the transition function and currenct configuration C, there can only be one next configuration C'. The entire description of the Turing machine at some moment is just the contents of the tape, the current state, and the position of the tape head. We encode these together as one string as follows.

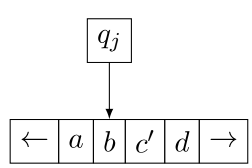
Consider the position below



If $\delta(q_i, c) = (q_j, c', \mathrm{R})$, then $abq_icd \vdash abc'q_jd$



If $\delta(q_i, c) = (q_j, c', \mathrm{L})$, then $abq_icd \vdash aq_jbc'd$



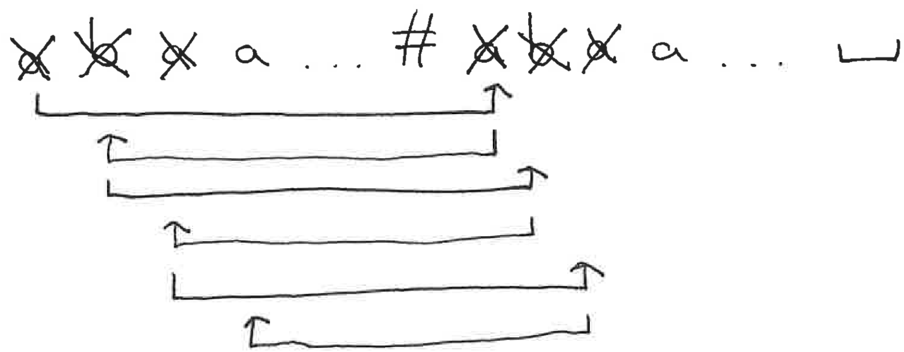
$\star$ NOTE: $\vdash$ means yields

Notice that for sequential configurations, only a small local portion of the configuration is changed. The initial configuration of any Turing machine is always $q_0 w_1 \dots w_n$. We may define the accepting and rejecting configurations appropriately, if they contain the accept or reject state. We say a Turing machine accepts $w_1 \dots w_n$ if there exists a sequence of configurations $C_0, C_1, C_2 ..., C_k$

$C_0 = q_0 w_1 \dots w_n$
$C_i \vdash C_{i+1}$
C_k is accepting

1.2 Create a Turing Machine for $\{w\#w \mid w \in \Sigma^*\}$

Lets give a Turing machine to decide this language. Before we start drawing states, lets consider a way to decide this language from a high level, pseudocode perspective. The idea is we check if it is of the correct form letter by letter, one at a time, resetting ourselves for each correctly. Then if all the letters are the same, we accept.



1.2.1 Pseudocode

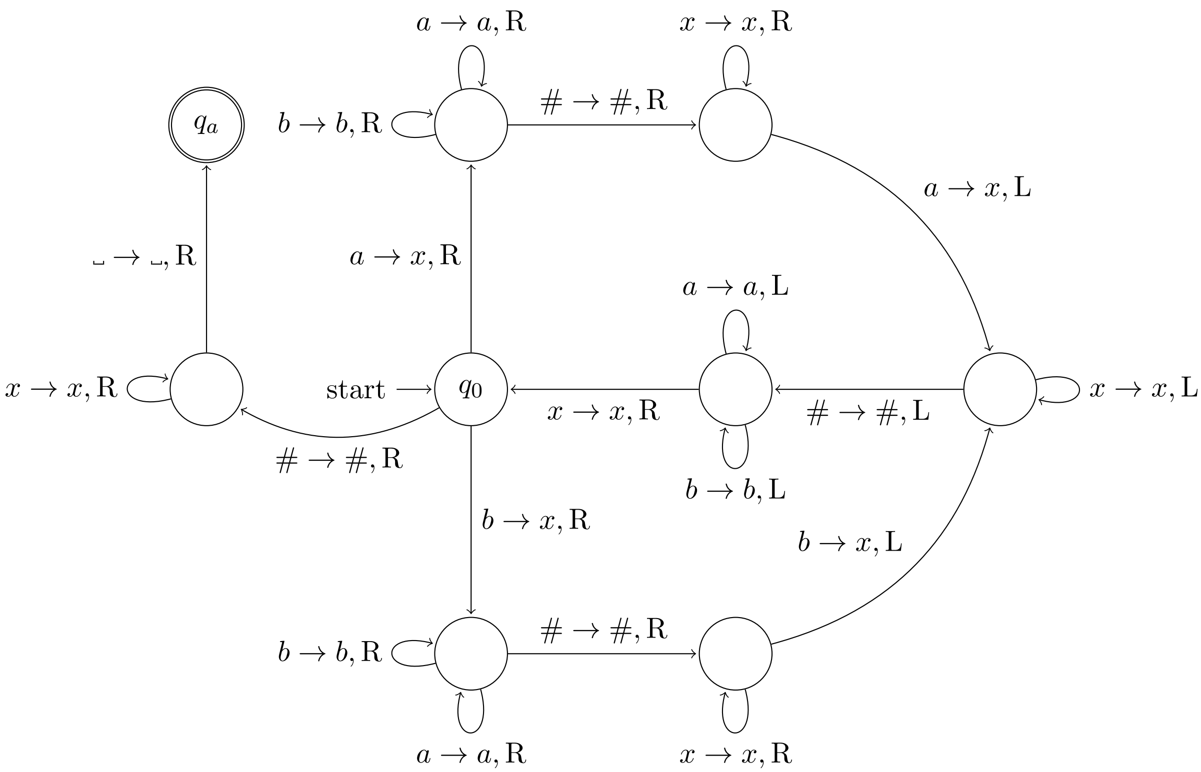
M on input w:

1. mark and remember symbol, keep track in states
2. loop right until #
3. loop past any marked marked
4. if next unmarked is what was first seen:
   (a) mark
5. loop left until #
6. loop past any unmarked
7. reset to first unmarked (repeat from step 1)

8. if no symbols besides x before hash remain:
   (a) if no symbols besides x after hash remain:
      i. accept
9. reject

1.2.2 State Diagram

We can attempt to give a state diagram for this language. We will omit q_r, as it gets too messy. Undefined transitions in this diagram, you should understand to mean implicit rejection. We encode the transitions to be of the form `read,write` $\rightarrow$ `move`. One transition performs all three operations. You read the symbol under the tape head, then conditionally write, move the tape head left or right one cell, and change states.

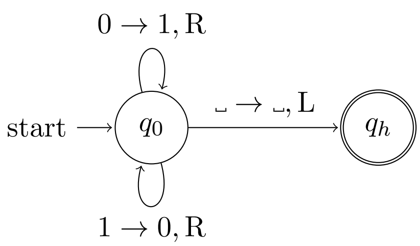


Note how we have two branches from q_0, if we mark a or b. We use the states of the machine to keep track of this single symbol we have seen. We may use the states to keep track of finitely many things, like registers. We may also keep track of other things on the arbitrarily large tape. Although this example only used the same space as the input, the machine may use as much tape as it wants.

2 Computation

Unlike previous models, Turing machines do more than just decide languages. They can also compute! A function $f : \Sigma^* \to \Sigma^*$ is **<u>Turing-Computable</u>** (or just computable) if there exists a Turing Machine on all inputs w halts with only $f(w)$ on its tape. Instead of an accepting and rejecting state, we simply have a halt state q_h. Lets give several Turing machines to compute some common functions.

2.1 f : Bit Flips

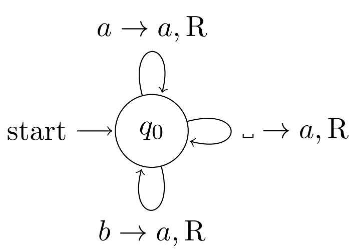


Example: Configurations for input 101

- $q_0 101$
- $0 q_0 01$
- $01 q_0 1$
- $010 q_0 \sqcup$
- $01 q_h 0 \sqcup$

We halted in four steps. Note that when we transition to the halting state, it doesn't really matter whether we move left or right. It doesn't matter where the tape head is when we halt, the resulting output is 010.

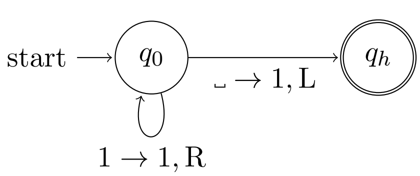
2.2 Turing Machines do NOT have to halt



The Turing Machine that writes a in every cell forever. Regardless of what it sees on its tape, its forever will march right, never stopping. It computes no function, but the

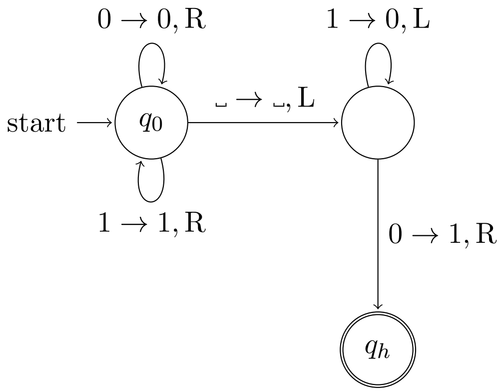
2.3 Successor Function: $S(x) = x + 1$

2.3.1 Unary Succession



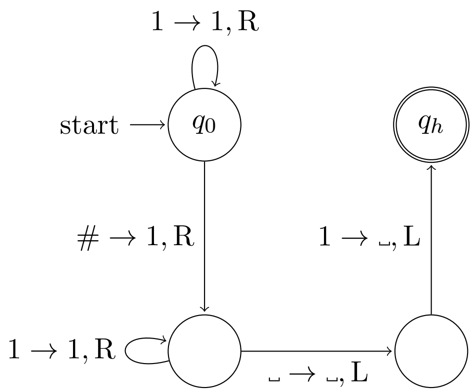
We begin with 1^x on the tape Halt with 1^{x+1}. The idea is we just loop to the end and toss a stick onto the pile.

2.3.2 Binary



For simplicity suppose the input is always given with a leading zero. We begin on the left on the input, so first we move all the way to the right. When adding 1 in binary, we loop from the right zeroing out all 1s until we find the first 0 and make it a 1.

2.4 Addition of 2 Numbers



$\mathrm{add}(x, y) = x + y$. Lets begin with $1^x\#1^y$ on the tape. We want to halt with 1^{x+y} on the tape. The simplest idea is to replace the # with a 1 and remove the last 1 at the end. It would be challenging, but convince yourself you could give a Turing machine for addition in binary.

3 Decidability vs Recognizability

Recall, a function $f : \Sigma^* \to \Sigma^*$ is **Turing-Computable** (or computable) if there exists a Turing Machine for all inputs w, which when initialized with w on the tape, halts with $f(w)$ on its tape.

Additionally, a language, L, is **Turing-Decidable** (or just decidable or recursive) if there exists a Turing Machine, M, such that

- $w \in L \iff M$ accepts w
- $w \notin L \iff M$ rejects w

Notice that for every input, a decide always halts. A language, L, is **Turing-Recognizable** (or just recognizable or recursively-enumerable) if there exists a Turing Machine, M, such that

- $w \in L \iff M$ accepts w
- $w \notin L \iff M$ rejects w or gets stuck in a loop

We allow a recognizer to loop on some inputs. If the answer is supposed to be yes, it must always halt and accept. If the answer is suppose to be no, then it may halt and reject, or loop. It is clear to see that every decidable language is also recognizable, but is every recognizable language also decidable? We shall see.