a few NP-complete graph problems.

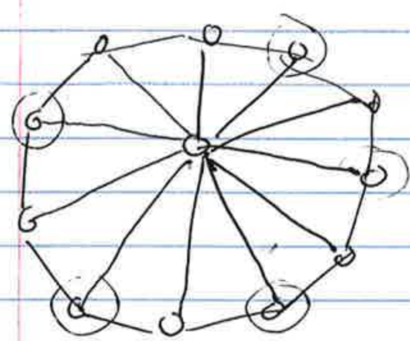
I want you to get into the habit of proving reductions quickly. Last time we discussed SAT and 3SAT. Today we are going to prove all of the following reductions

$$\text{SAT} \rightarrow \text{3SAT} \rightarrow \text{Circuit SAT}$$
$$\text{3SAT} \rightarrow \text{INDSET}$$
$$\text{INDSET} \rightarrow \text{VERTEXCOVER}$$
$$\text{INDSET} \rightarrow \text{CLIQUE}$$

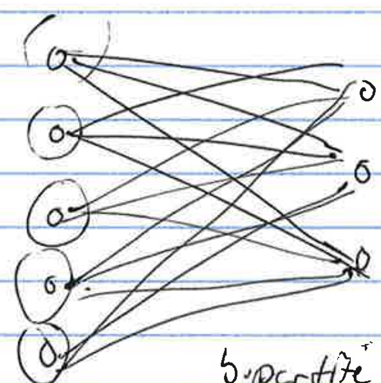
☆ For circuit SAT, see the previous notes.

$\text{INDSET} = \{ \langle G, g \rangle \mid G \text{ has an independent set of size } (\geq)\ g \}$

a set of vertices is independent if there is no edge between any two. Its easy to solve in some simple cases



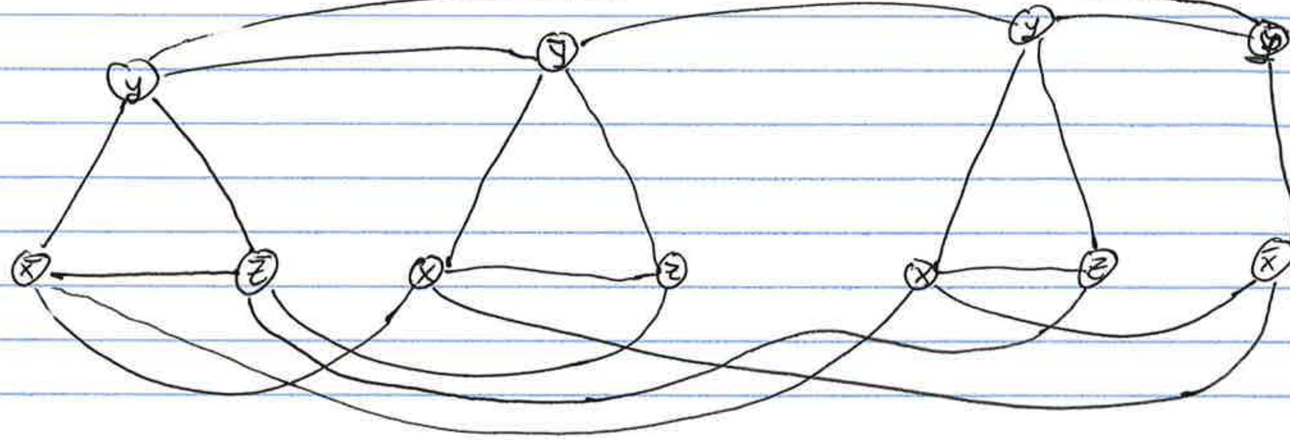
wheel graphs



bipartite graph.

However, in general, INDSET is NP-complete.
first we prove INDSET $\in$ NP.
our Verifier takes as input a problem $\langle G, g \rangle$ and witness $\langle v_1 \ldots v_k \rangle$
$V(\langle G, g \rangle, \langle v_1 \ldots v_k \rangle)$ ~~and~~ checks if $k \geq g$. Then checks if each pair $v_i, v_j \in v_1 \ldots v_k$ has no edge in G. This takes polytime so INDSET $\in$ NP.

$(\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y})$



if $\phi \in 3SAT$, then $\exists$ a satisfying assignment. Select the appropriate vertices as part of the independent set. Since one vertice of each a clause will be selected, the size of the independent set $= g$ so $\phi \in 3SAT \Rightarrow \langle G, g \rangle \in INDSET$.

If $\langle G, g \rangle \in INDSET$, $g =$ number of triangles, there is an independent set of size there. Since you cannot pick two (or more) from any triangle, only one, this independent set must have one vertex exactly in each triangle. This vertex corresponds to the assignment ~~of~~ each clause in the formula ϕ, so we see $\langle G, g \rangle \in INDSET \Rightarrow \phi \in 3SAT$.

Also note f is computable obviously in polytime. building G takes linear time to build the triangles, then n^2 worst case for the other pairs of edges.

we conclude since $INDSET \in NP$ and ~~3SAT~~

$3SAT \leq_p INDSET$ that $INDSET$ is NP-complete.

Now that we have one graph problem which is NP-complete, it will make ~~the~~ ~~[illegible]~~ more graph NP-complete problems easier to prove.

INDSET is a graph problem. All our other reductions have been from boolean formula problems How can we convert a CNF-formula to a graph? seems non trivial.

When you have a satisfying assignment of a formula, you cannot select x and $\bar{x}$ simultaneously. for example.

$$(a \vee x \vee b) \wedge (c \vee \bar{x} \vee d)$$

choosing x in the first clause turns off $\bar{x}$ in the second. We want to do something similiar. Choosing a vertex to be part of ~~a~~ a candidate independent set turns off its neighbors from being included in the set.

we build a polytime computable f such that

$$\phi \in 3SAT \iff \langle G, g \rangle \in INDSET$$

$f(\phi) = \langle G, g \rangle$.

f on input ϕ, a formula in 3CNF
create one "triangle" per clause
where every vertex in the clause is unique
vertices labeled by the clauses literals.
If x a vertex in a triangle, Put an edge
to $\bar{x}$ in all other triangles.
set g to be number of clauses
return $\langle G, g \rangle$

Clique. Notice that if a set of vertices are independent, then they share no edges. In the opposite, the complement of the graph. This set would share all edges.

a clique is a graph ~~at~~ where every vertex is connected

etc.

K_2 K_3 K_4 K_5 K_6

specifically, a clique is a connected subgraph, rather than some graph itself.

a clique in a graph is an independent set in the graph complement. For $G = (V, E)$, Define $\overline{G} = (V, \overline{E})$ if $e \in E$ then $e \notin \overline{E}$. If $e \notin E$, then put $e \in \overline{E}$, and so on.

~~CLIQUE $\in$ NP or verifier~~

CLIQUE $= \{ \langle G, g \rangle \mid G$ has a clique of size $\geq g \}$ is NP complete. To prove CLIQUE $\in$ NP, our polytime verifier V takes as input $V(\langle G, g \rangle, \langle v_1 \ldots v_k \rangle)$. It checks that $v_1 \ldots v_k$ has size $\geq g$. Next it checks if $\exists$ an edge between every pair $v_i, v_j \in v_1 \ldots v_k$. This $\Rightarrow$ our witness, $v_1 \ldots v_k$ is a clique. V takes polytime so CLIQUE $\in$ NP.

Now we reduce not from 3SAT, but from INDSET. We prove INDSET $\le_p$ CLIQUE. Let ~~$f: \langle G, g \rangle$~~ $f: \langle G, g \rangle \to \langle \bar{G}, g \rangle$. We prove

$$\underset{\langle G,g \rangle}{x} \in \text{INDSET} \iff \underset{\langle \bar{G},g \rangle}{f(x)} \in \text{CLIQUE}.$$

If $\langle G, g \rangle$ in INDSET, then $\exists$ a selection of the vertices of size g with no edges between them. Then if $e \notin E \Rightarrow e \in \bar{E}$, so those same vertices in $\bar{G}$ share all edges. That is the definition of a clique, so $\bar{G}$ has this clique of size g. So $\langle G, g \rangle \in$ INDSET $\Rightarrow \langle \bar{G}, g \rangle \in$ CLIQUE.

Note since $\bar{\bar{G}} = G$, this is a repeat, here we get the reverse argument for free. If $\langle \bar{G}, g \rangle \in$ CLIQUE then the clique of size g in $\bar{G}$ is an independent set of size g in ~~$\langle G, g \rangle$~~ G so $\langle G, g \rangle \in$ INDSET.

This transformation of flipping edges takes polynomial time so since CLIQUE $\in$ NP, INDSET is NP-complete and INDSET $\le_p$ CLIQUE, we see CLIQUE is NP-complete.

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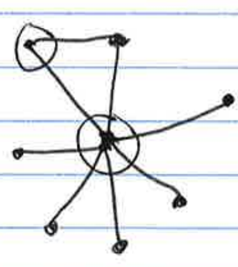
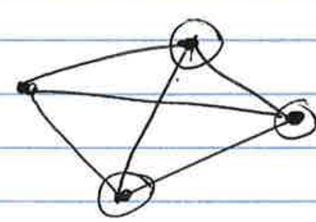
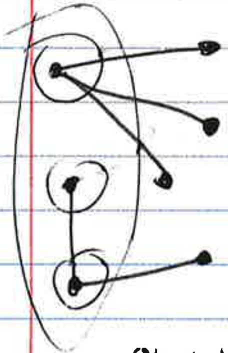
a vertex cover is a selection of vertices such that every edge shares one end in the cover.



a vertex cover



notice, the opposite of a vertex cover can have no edges. thats just an independent set! so a set ~~S is a~~ $S \subseteq V$ is a vertex cover if $\bar{S} = V \setminus S$ is an independent set.



VERTEX-COVER $= \{ \langle G, k \rangle \mid G$ has a vertex cover of size $k \}$ is NP-complete.

first we show VERTEX-COVER $\in$ NP

our verifier V on input $V(\langle G,k \rangle, v_1 \dots v_k)$ checks for each edge $e \in E$ if one endpoint is in $v_1 \dots v_k$. If it is then it accepts. obviously polytime.

Now we reduce from INDSET to VERTEXCOVER. our transformation is $f : \langle G, g \rangle \to \langle G, |V| - g \rangle$.

if $\langle G, g \rangle \in$ INDSET. $\exists$ a selection of V, say S with no edges between them. then $\bar{S} = V - S$ is a set of vertices where all edges have to touch. If no edge ~~touches any~~ has both ends in S, every edge has one end in $V \setminus S$. So $\bar{S}$ is a vertex cover of size $|V| - g$ so $\langle G, |V| - g \rangle \in$ VERTEX COVER.

Similarly if $\langle G, k \rangle \in$ VERTEX-COVER then $\exists$ a selection $S \subseteq V$ where every edge has atleast one end in S. Then since S is a vertex cover, $\bar{S} = V - S$ has no edges, so $\bar{S}$ is an independent set of size $|V| - k = |V| - (|V| - g) = g$. so $\langle G, g \rangle \in$ INDSET and we see its an $\Longleftrightarrow$. This transformation is obviously polytime so since VERTEXSET $\in$ NP and INDSET $\leq_p$ VERTEXSET we see VERTEXSET is NP-complete.